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# On A Certain Class Of Starlike Functions. II

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ABSTRACT. There are many classes of starlike functions in the unit disc $U=\{z: |z|< 1\}$. In this paper we consider a class $S_p^*(\alpha,\beta,\gamma,A,B)$ of starlike functions of the form $f(z)=z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} (p \in N)$ in the unit disc $U$ and satisfying the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{(B-A)\gamma(\frac{zf'(z)}{f(z)} -\alpha) + A(\frac{zf'(z)}{f(z)} - 1)} \right| < \beta$$

for some $\alpha (0\leq\alpha<1)$, $\beta(0<\beta\leq1)$, $\gamma(0<\gamma\leq1)$ and $-1\leq A<B\leq1$, $0<B\leq1$.

It is the purpose of this paper to show a representation formula, a distortion theorem and a sufficient condition for the class $S_p^*(\alpha,\beta,\gamma,A,B)$. Moreover we give the radius of convexity for functions in the class $S_p^*(\alpha,\beta,\gamma,A,B)$.

## 1. Introduction.

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $U=\{z: |z| <1\}$. A

A function $f(z) \in S$ is said to be starlike of order $\alpha$ $(o\leq\alpha<1)$ in the unit disc U if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$$

for some $\alpha$, and for all $z \in U$. And the above condition is equivalent to

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\left(\frac{zf'(z)}{f(z)} - \alpha\right) - \left(\frac{zf'(z)}{f(z)} - 1\right)} \right| < 1 .$$

In this paper, we consider the class $S_p^*(\alpha,\beta,\gamma,A,B)$ of functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N= \{1,2,\ldots\})$$

which are analytic and starlike in the unit disc U satisfying the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)} - 1}{(B-A)\gamma\left(\dfrac{zf'(z)}{f(z)} - \alpha\right) + A\left(\dfrac{zf'(z)}{f(z)} - 1\right)} \right| < \beta \tag{1.1}$$

for some $\alpha\,(0 \leq \alpha < 1)$, $\beta\,(0 < \beta \leq 1)$, $\gamma\,(0 < \gamma \leq 1)$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, and for all $z \in U$. The classes $S_1^*(\alpha, 1, \frac{1}{2}, -1, 1)$, $S_1^*(0, 1, \frac{2\gamma - 1}{2\gamma}, -1, 1)\,(\gamma > \frac{1}{2})$, $S_1^*(\frac{1-\gamma}{1+\gamma}, 1, \frac{1+\gamma}{2}, -1, 1)\,(0 < \gamma \leq 1)$, $S_1^*(1-\alpha, 1, \frac{1}{2}, -1, 1)$, $S_1^*(\alpha, 1, \gamma, -1, 1)$ and $S_1^*(\alpha, \beta, \gamma, A, B)$ were studied by McCarty [6], Singh [13,14], Padmanabhan [12], Eenigenburg[3], Juneja and Mogra [4] and Aouf and Nunokawa [1], respectively. Also in [8,9,10,11] Owa showed some results for functions in the class $S_1^*(\alpha, \beta, \gamma, -1, 1)$.

2. A representation formula.

First of all, we require the following lemma.

Lemma 1. Let a function

$$H(z) = 1 + b_p z^p + b_{p+1} z^{p+1} + \ldots\ldots \qquad (p \in N)$$

be analytic in the unit disc U. Then H(z) satisfies the condition

$$\left| \frac{H(z) - 1}{(B-A)\gamma(H(z) - \alpha) + A(H(z) - 1)} \right| < \beta \qquad (z \in U)$$

for some $\alpha\,(0 \leq \alpha < 1)$, $\beta\,(0 < \beta \leq 1)$, $\gamma\,(0 < \gamma \leq 1)$ and $-1 < A \leq B \leq 1$, $0 < B \leq 1$, if and only if there exists an analytic function $\phi(z)$ in the unit disc U such that $|\phi(z)| \leq \beta$ for $z \in U$ and

$$H(z) = \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)} .$$

Proof. We use a method by Padmanabhan [12]. Assume that a function

$$H(z) = 1+b_p z^p + b_{p+1} z^{p+1} + \ldots\ldots \qquad (p \in N)$$

satisfies the condition

$$\left| \frac{H(z)-1}{(B-A)\gamma(H(z)-\alpha)+A(H(z)-1)} \right| < \beta$$

for some $\alpha\,(o \leq \alpha < 1)$, $\beta\,(o < \beta \leq 1)$, $\gamma\,(o < \gamma \leq 1)$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$. Let

$$z^{p-1}h(z) = \frac{1-H(z)}{(B-A)\gamma(H(z)-\alpha)+A(H(z)-1)} .$$

Then we see that the function h(z) is analytic in the unit disc U, and $|h(z)| < \beta$ for $z \in U$ and h(o)=o. Accordingly, by Schwarz's Lemma [7], we have $h(z)=z\phi(z)$, where $\phi(z)$ is an analytic function in the unit disc U and satisfies $|\phi(z)| \leq \beta$ for $z \in U$. Thus we get

$$H(z) = \frac{1+[(B-A)\alpha\gamma+A]z^{p-1}h(z)}{1+[(B-A)\gamma+A]z^{p-1}h(z)}$$

$$= \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)} .$$

Conversely, if

$$H(z) = \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)}$$

and $|\phi(z)| \leqslant \beta$ for $z \in U$, then $H(z)$ is an analytic in the unit disc $U$. Further, since $|z^p\phi(z)| \leqslant \beta |z|^p < \beta$ for $z \in U$, we get

$$\left| \frac{H(z)-1}{(B-A)\gamma(H(z)-\alpha) + A(H(z) - 1)} \right| = |z^p\phi(z)| < \beta$$

for $z \in U$. Hence we have the lemma.

Theorem 1. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be analytic in the unit disc $U$. Then the function $f(z)$ belongs to the class $S_p^*(\alpha,\beta,\gamma,A,B)$ if and only if

$$f(z)=z \exp\left\{-(B-A)\gamma(1-\alpha) \int_0^z \frac{t^{p-1}\phi(t)}{1+[(B-A)\gamma+A]t^p\phi(t)} dt\right\}, \quad (2.1)$$

where $\phi(z)$ is analytic in the unit disc $U$ and satisfies $|\phi(z)| \leqslant \beta$ for $z \in U$.

Proof. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

belong to the class $S_p^*(\alpha,\beta,\gamma,A,B)$. Then, since the function $f(z)$ satisfies the condition (1.1), we can write

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)}$$

by using Lemma 1. Hence we get

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = -\frac{(B-A)\gamma(1-\alpha)z^{p-1}\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)} .$$

On integrating both sides of the above equality from o to z, we obtain the representation formula (2.1).

On the other hand, if $f(z)$ has the representation (2.1), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)}$$

holds with $\phi(z)$ as in Lemma 1. Therefore we see that $f(z)$ is in the class $S_p^*(\alpha,\beta,\gamma,A,B)$ by using Lemma 1.

3. A distortion theorem.

Lemma 2. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

belong to the class $S_p^*(\alpha,\beta,\gamma,A,B)$. Then we have

$$\frac{1+[(B-A)\alpha\gamma+A]\beta|z|^p}{1+[(B-A)\gamma+A]\beta|z|^p} \leqslant \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \leqslant \frac{1-[(B-A)\alpha\gamma+A]\beta|z|^p}{1-[(B-A)\gamma+A]\beta|z|^p}$$

for $z \in U$.

<u>Proof</u>. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be in the class $S_p^*(\alpha,\beta,\gamma,A,B)$. Then, by using Schwarz's Lemma [7], the condition (1.1) implies that $\frac{zf'(z)}{f(z)}$ assumes values lying in the disc obtained by taking the line segment joining two poits

$$\frac{1+[(B-A)\alpha\gamma+A]\beta|z|^p}{1+[(B-A)\gamma+A]\beta|z|^p}$$

and

$$\frac{1-[(B-A)\alpha\gamma+A]\beta|z|^p}{1-[(B-A)\gamma+A]\beta|z|^p}$$

as diameter. Consequently we have the lemma.

Theorem 2. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be analytic in the unit disc U and suppose $f(z) \in S_p^*(\alpha, \beta, \gamma, A, B)$. Then we have

$$|f(z)| \geqslant \frac{|z|}{\left\{1+[(B-A)\gamma+A]\beta|z|^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}}$$

and

$$|f(z)| \leqslant \frac{|z|}{\left\{1-[(B-A)\gamma+A]\beta|z|^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}}$$

for $o \leqslant \alpha < 1$, $o < \beta \leqslant 1$, $o < \gamma \leqslant 1$, $\gamma \neq \frac{-A}{(B-A)\gamma}$, $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$ and $z \in U$. Moreover, for $o \leqslant \alpha < 1$, $o < \beta \leqslant 1$, $\gamma = \frac{-A}{(B-A)}$, $-1 \leqslant A < B \leqslant 1$, $o < B \leqslant 1$ and $z \in U$, we have

$$|z| \exp\left\{\frac{-A\beta(\alpha-1)}{p}|z|^p\right\} \leqslant |f(z)| \leqslant |z| \exp\left\{\frac{-A\beta(1-\alpha)}{p}|z|^p\right\}.$$

Proof. Since the function $f(z)$ is in the class $S_p^*(\alpha, \beta, \gamma, A, B)$, we have

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)},$$

where $\phi(z)$ is an analytic function in the unit disc $U$ and $|\phi(z)| \leq \beta$ for $z \in U$. Consequently we obtain

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{(B-A)\gamma(1-\alpha)z^{p-1}\phi(z)}{1+[(B-A)\gamma+A]z^{p-1}\phi(z)}. \tag{3.1}$$

On integrating both sides of (3.1) from o to z and taking real parts of both sides of the resulting equation, we have

$$\log\left|\frac{f(z)}{z}\right| = \mathrm{Re}\left\{\log\left(\frac{f(z)}{z}\right)\right\} = \mathrm{Re}\int_o^z \left\{\frac{f'(t)}{f(t)} - \frac{1}{t}\right\}dt$$

$$= \mathrm{Re}\int_o^z \frac{-(B-A)\gamma(1-\alpha)t^{p-1}\phi(t)}{1+[(B-A)\gamma+A]t^p\phi(t)}\,dt$$

$$\leq \int_o^{|z|} \frac{(B-A)\gamma(1-\alpha)\left|\phi(te^{i\theta})\right|t^{p-1}}{\left|1+[(B-A)\gamma+A]t^p e^{ip\theta}\phi(te^{i\theta})\right|}\,dt.$$

Hence

$$\log\left|\frac{f(z)}{z}\right| \leq \int_o^{|z|} \frac{(B-A)\beta\gamma(1-\alpha)t^{p-1}}{1-[(B-A)\gamma+A]\beta t^p}\,dt$$

$$= -\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}\log\left\{1-[(B-A)\gamma+A]\beta|z|^p\right\}$$

$$= -\log\left\{1-[(B-A)\gamma+A]\beta|z|^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}$$

for $o \leq \alpha < 1$, $o < \beta \leq 1$, $o < \gamma \leq 1$, $\gamma \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$ and $z \in U$. Furthermore, for $o \leq \alpha < 1$, $o < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$

and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, we obtain

$$\log \left| \frac{f(z)}{z} \right| \leq -A\beta(1-\alpha) \int_0^{|z|} t^{p-1}\, dt$$

$$= \frac{-A\beta(1-\alpha)}{p} |z|^p .$$

Therefore we see that

$$|f(z)| \leq \frac{|z|}{\left\{1-[(B-A)\gamma+A]\beta|z|^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$, and that

$$|f(z)| \leq |z| \exp\left\{ \frac{-A\beta(1-\alpha)}{p} |z|^p \right\}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $0 < B \leq 1$.

On the other hand, by using Lemma 2, we get

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{1+[(B-A)\alpha\gamma+A]\beta|z|^p}{1+[(B-A)\gamma+A]\beta|z|^p}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $z \in U$. From this, we obtain

$$r\,\mathrm{Re}\left\{\frac{\partial}{\partial r}\left(\log\frac{f(z)}{z}\right)\right\} = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - 1\right\}$$

$$\geqslant \frac{1+[(B-A)\alpha\gamma+A]\beta r^p}{1+[(B-A)\gamma+A]\beta r^p} - 1$$

$$= -\frac{(B-A)\beta\gamma(1-\alpha)r^p}{1+[(B-A)\gamma+A]\beta r^p}$$

for $|z| = r$. Thus we see that

$$\log\left|\frac{f(z)}{z}\right| = \mathrm{Re}\left\{\log\frac{f(z)}{z}\right\}$$

$$\geqslant \int_0^r \frac{-(B-A)\beta\gamma(1-\alpha)t^{p-1}}{1+[(B-A)\gamma+A]\beta t^{p-1}}dt.$$

Therefore we have

$$\log\left|\frac{f(z)}{z}\right| \geqslant -\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}\log\left\{1+[(B-A)\gamma+A]\beta r^p\right\}$$

$$= -\log\left\{1+[(B-A)\gamma+A]\beta r^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}$$

for $0\leqslant\alpha<1$, $0<\beta\leqslant1$, $0<\gamma\leqslant1$, $\gamma\neq\frac{-A}{(B-A)}$ and $-1\leqslant A<B\leqslant1$, $0<B\leqslant1$ and

$$\log\left|\frac{f(z)}{z}\right| \geqslant \frac{-A\beta(1-\alpha)}{p}\, r^p$$

for $o \leq \alpha < 1$, $o < \beta \leq 1$, $o < \gamma \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$.
Consequently

$$|f(z)| \geq \frac{|z|}{\left\{1+[(B-A)\gamma + A]\beta\,|z|^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}}$$

for $o \leq \alpha < 1$, $o < \beta \leq 1$, $o < \gamma \leq 1$, $\gamma \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$,
and

$$|f(z)| \geq |z| \exp\left\{\frac{-A\beta(\alpha-1)}{p}\,|z|^p\right\}$$

for $o \leq \alpha < 1$, $o < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$. Finally, for equality, we may take

$$f(z) = \frac{z}{\left\{1-[(B-A)\gamma+A]\beta z^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}}$$

for $o \leq \alpha < 1$, $o < \beta \leq 1$, $o < \gamma \leq 1$, $\gamma \neq \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$,
and

$$f(z) = z \exp\left\{\frac{-A\beta(1-\alpha)}{p}\, z^p\right\}$$

for $o \leq \alpha < 1$, $o < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$.

4. A sufficient condition for the class $S_p^*(\alpha,\beta,\gamma,A,B)$.

Theorem 3. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in N)$$

be analytic in the unit disc U. If we have

$$\sum_{n=1}^{\infty}\Big\{(p+n-1)+\beta(-A(p+n+1)-(B-A)\gamma p-(B-A)\gamma n$$

$$-(B-A)\alpha\gamma\Big\}|a_{p+n}| \leq (B-A)\beta\gamma(1-\alpha) \tag{4.1}$$

for some $\alpha(o\leq\alpha<1)$, $\beta(o<\beta\leq 1)$, $\gamma(o<\gamma\leq \frac{-A}{(B-A)})$ and $-1\leq A<B\leq 1$, $o<B\leq 1$, then the function f(z) belongs to the class $S_p^*(\alpha,\beta,\gamma,A,B)$.

Proof. We employ the technique used by Clunie and Keogh [2]. We assume that the condition (4.1) holds. Then we obtain

$$|zf'(z)-f(z)| - \beta|(B-A)\gamma(zf'(z)-\alpha f(z))+A(zf'(z)-f(z))|$$

$$=\left|\sum_{n=1}^{\infty}(p+n-1)a_{p+n}z^{p+n}\right| - \beta\Big|(B-A)\gamma(1-\alpha)z +$$

$$\sum_{n=1}^{\infty}(-A-(B-A)\alpha\gamma)a_{p+n}z^{p+n} - \sum_{n=1}^{\infty}(-A-(B-A)\gamma)(p+n)a_{p+n}z^{p+n}\Big|$$

$$\leq \sum_{n=1}^{\infty}(p+n-1)|a_{p+n}||z|^{p+n} -$$

$$\Big\{\Big|(B-A)\beta\gamma(1-\alpha)z+\sum_{n=1}^{\infty}(-A-(B-A)\alpha\gamma)\beta\, a_{p+n}z^{p+n}\Big|$$

$$-\sum_{n=1}^{\infty}(-A-(B-A)\gamma)\beta(p+n)\,|a_{p+n}||z|^{p+n}\Big\}$$

$$\leqslant \sum_{n=1}^{\infty}(p+n-1)\,|a_{p+n}||z|^{p+n} -$$

$$\Big\{(B-A)\beta\gamma(1-\alpha)\,|z| - \sum_{n=1}^{\infty}(-A-(B-A)\alpha\gamma)\beta|a_{p+n}||z|^{p+n}$$

$$-\sum_{n=1}^{\infty}(-A-(B-A)\gamma)\beta\,(p+n)\,|a_{p+n}||z|^{p+n}\Big\}$$

$$\leqslant [\sum_{n=1}^{\infty}\Big\{(p+n-1)+\beta(-A(p+n+1)-(B-A)\gamma p$$

$$-(B-A)\alpha\gamma-(B-A)\gamma n\Big\}\cdot|a_{p+n}|-(B-A)\beta\gamma(1-\alpha)]\,|z| \leqslant 0$$

for $0\leqslant\alpha<1$, $0<\beta\leqslant 1$, $0<\gamma\leqslant\frac{-A}{(B-A)}$, $-1\leqslant A<B\leqslant 1$, $0<B\leqslant 1$ and $z\in U$. Hence, by the maximum modulus theorem f(z) is in the class $S_p^*(\alpha,\beta,\gamma,A,B)$.

5. The radius of convexity for functions in the class $S_p^*(\alpha,\beta,\gamma,A,B)$.

Theorem 4. Let a function

$$f(z) = z + \sum_{n=1}^{\infty} a_{p+n}z^{p+n} \qquad (p\in N)$$

be in the class $S_p^*(\alpha,\beta,\gamma,A,B)$ with $0\leq\alpha<1$, $0<\beta\leq 1$, $0<\gamma\leq \frac{-A}{(B-A)}$ and $-1\leq A<B\leq 1$, $0<B\leq 1$. Then the function $f(z)$ maps

$$|z| < \left(\frac{-A+(\frac{B-A}{2})\gamma(1-\alpha)-\sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)]}}{\beta}\right)^{\frac{1}{p}}$$

on to a convex domain if

$$\left\{[-A+(\tfrac{B-A}{2})\gamma(1-\alpha)]-\sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]}\right\}.$$

$$.[(\tfrac{B-A}{2})\gamma(1-\alpha)+\sqrt{(B-A)\gamma(1-\alpha)[-A+(\tfrac{B-A}{2})\gamma(1-\alpha)]}]^p$$

$$\leq\beta\left(\sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]}\right)^p$$

$$\leq\left(\sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]}\right)^p.$$

This result is sharp.

Proof. We employ the technique used by Lakshminarasimhan [5]. Since $f(z)$ belongs to the class $S_p^*(\alpha,\beta,\gamma,A,B)$, by Theorem 1, we get

$$\frac{zf'(z)}{f(z)} = \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)}, \tag{5.1}$$

where $\phi(z)$ is analytic in the unit disc $U$ and satisfies $|\phi(z)|\leq\beta$ for $z\in U$. On differentiating both sides of (5.1) with respect to z logarithmically, we have

$$1+\frac{zf''(z)}{f'(z)} = \frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)}$$

$$-\frac{(B-A)\gamma(1-\alpha)\{pz^p\phi(z)+z^{p+1}\phi'(z)\}}{\{1+[(B-A)\gamma+A]z^p\phi(z)\}\{1+[(B-A)\alpha\gamma+A]z^p\phi(z)\}}\,.$$

Furthermore we have

$$\left|\frac{\phi'(z)}{\beta}\right| \leqslant \frac{1-\left|\frac{\phi(z)}{\beta}\right|^2}{1-|z|^2} \tag{5.2}$$

for the analytic function $\phi(z)$ in the unit disc U. Now, since

$$\mathrm{Re}\left\{\frac{1+[(B-A)\alpha\gamma+A]z^p\phi(z)}{1+[(B-A)\gamma+A]z^p\phi(z)}\right\}$$

$$=\frac{1+[(B-A)\alpha\gamma+A][(B-A)\gamma+A]\left|z^p\phi(z)\right|^2+[(B-A)\alpha\gamma+(B-A)\gamma+2A]\,\mathrm{Re}\,(z^p\phi(z))}{\left|1+[(B-A)\gamma+A]z^p\phi(z)\right|^2}$$

$$\geqslant\frac{\{1+[(B-A)\alpha\gamma+A]\,|z^p\phi(z)|\}\{1+[(B-A)\gamma+A]\,|z^p\phi(z)|\}}{\left|1+[(B-A)\gamma+A]z^p\phi(z)\right|^2}$$

$$\geqslant\frac{1+[(B-A)\alpha\gamma+A]\,|z^p\phi(z)|}{1+[(B-A)\gamma+A]\,|z^p\phi(z)|}$$

and

$$\operatorname{Re}\left\{\frac{pz^p\phi(z)+z^{p+1}\phi'(z)}{\{1+[(B-A)\gamma+A]\,|z^p\phi(z)|\}\{1+[(B-A)\alpha\gamma+A]\,|z^p\phi(z)|\}}\right\}$$

$$\leqslant \frac{p\,|z^p\phi(z)|+|z^{p+1}\phi'(z)|}{\{1+[(B-A)\gamma+A]\,|z^p\phi(z)|\}\{1+[(B-A)\alpha\gamma+A]\,|z^p\phi(z)|\}}$$

$$\leqslant \frac{p\,|z^p\phi(z)|+|z^{p+1}\phi'(z)|}{\{1+[(B-A)\gamma+A]\,|z^p\phi(z)|\}^2},$$

we obtain

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}\geqslant \frac{1+[(B-A)\alpha\gamma+A]\,|z^p\phi(z)|}{1+[(B-A)\gamma+A]\,|z^p\phi(z)|}$$

$$-\frac{(B-A)\gamma(1-\alpha)\{p\,|z^p\phi(z)|+|z^{p+1}\phi'(z)|\}}{\{1+[(B-A)\gamma+A]\,|z^p\phi(z)|\}^2}.$$

If we assume that

$$1+A^2\,|z^p\phi(z)|^2+[2A-(B-A)\gamma(1-\alpha)p]\,|z^p\phi(z)|$$

$$-(B-A)\gamma(1-\alpha)\,|z^{p+1}\phi'(z)|>0, \tag{5.3}$$

then we have

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0.$$

Now, in virture of (5.2), the condition (5.3) will be satisfied

$$1+|z^p\phi(z)|^2 + [2A-(B-A)\gamma(1-\alpha)]\,|z^p\phi(z)| -$$

$$(B-A)\gamma(1-\alpha)\,|z|^{p+1}\,\frac{\beta - \frac{|\phi(z)|^2}{\beta}}{1-|z|^2} > o.$$

On putting $a = |z|$ and $t = |z^p\phi(z)|$, the above condition can be re-written as

$$(1-a^2)\left\{1+t^2+[2A-(B-A)\gamma(1-\alpha)]t\right\} -$$

$$(B-A)\gamma(1-\alpha)\left(\beta a^{p+1} - \frac{t^2}{\beta a^{p-1}}\right) > o,$$

that is,

$$t^2\left\{(1-a^2) + \frac{(B-A)\gamma(1-\alpha)}{\beta a^{p-1}}\right\} +$$

$$t(1-a^2)[2A-(B-A)\gamma(1-\alpha)] + 1-a^2-$$

$$(B-A)\beta\gamma(1-\alpha)a^{p+1} > o, \tag{5.4}$$

where $o<a<1$ and $o\leqslant t\leqslant \beta a^p$. If $G(t)$ denote the left hand side of (5.4), then we see that

$$G'(t) = 2t\left\{(1-a^2) + \frac{(B-A)\gamma(1-\alpha)}{\beta a^{p+1}}\right\} +$$

$$(1-a^2)[2A-(B-A)\gamma(1-\alpha)] = o$$

for

$$t = t_1 = \frac{\beta a^{p-1}(1-a^2)[-A+(\frac{B-A}{2})\gamma(1-\alpha)]}{\beta a^{p-1}(1-a^2) + (B-A)\gamma(1-\alpha)}$$

Moreover

$$G''(t) = 2\left\{(1-a^2) + \frac{(B-A)\gamma(1-\alpha)}{\beta a^{p-1}}\right\} > o,$$

because $o<a<1$. Now $t_1 - \beta a^p$ is positive and negative with

$$\beta a^{2p+1} - \beta a^{2p-1} - [-A+(\frac{B-A}{2})\gamma(1-\alpha)]a^{p+1}$$

$$-(B-A)\gamma(1-\alpha)a^p + [-A+(\frac{B-A}{2})\gamma(1-\alpha)]a^{p-1},$$

respectively. Let

$$E(a) = \beta a^{2p+1} - \beta a^{2p-1} - [-A+(\frac{B-A}{2})\gamma(1-\alpha)]a^{p+1}$$

$$-(B-A)\gamma(1-\alpha)a^p + [-A+(\frac{B-A}{2})\gamma(1-\alpha)]a^{p-1}$$

and let $a_o$ be the positive root of $E(a) = o$ lying in the open interval $(o,1)$. Then $E(a)$ is positive for $o<a<a_o$ and so $t_1 > \beta a^p$. Hence $G'(t)$ is negative for $o \leq t \leq \beta a^p$, $G(\beta a^p) < G(t)$ and the condition is satisfied if $G(\beta a^p) > o$. This is equivalent to

$$\beta^2 a^{2p}(1-a^2)-2\beta a^p(1-a^2)[-A+(\tfrac{B-A}{2})\gamma(1-\alpha)] + (1-a^2)>o,$$

that is,

$$(1-a^2)\left\{\beta^2 a^{2p}-2\beta[-A+(\tfrac{B-A}{2})\,\gamma\,(1-\alpha)]a^p + 1\right\} > o$$

which holds for

$$a<\left(\frac{[-A+(\frac{B-A}{2})\gamma(1-\alpha)]-\sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)]}}{\beta}\right)^{\frac{1}{p}}.$$

Further we can show that

$$a_o>\left(\frac{[-A+(\frac{B-A}{2})\gamma(1-\alpha)]-\sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)]}}{\beta}\right)^{\frac{1}{p}}$$

if

$$[-A+(\tfrac{B-A}{2})\,\gamma(1-\alpha)]-\sqrt{[-A\div 1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]}\leqslant\beta\leqslant 1.$$

The condition on $\beta$ implies that

$$\left(\frac{[-A+(\frac{B-A}{2})\gamma(1-\alpha)]-\sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)]}}{\beta}\right)^{\frac{1}{p}}<1,$$

and so

$$a_1 = \left(\frac{[-A+(\frac{B-A}{2})\gamma(1-\alpha)] - \sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)}}{\beta}\right)^{\frac{1}{p}} < a_o$$

if $E(a_1) > o$. Moreover $E(a_1) > o$ is satisfied if

$$\sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]} \; x$$

$$x \left\{[-A+(\tfrac{B-A}{2})\gamma(1-\alpha)] - \sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]}\right\}^{1-\frac{1}{p}} \beta^{\frac{2}{p}}$$

$$-(B-A)\gamma(1-\alpha) \left\{[-A+(\tfrac{B-A}{2})\gamma(1-\alpha)] - \right.$$

$$\left.\sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]}\right\} \beta^{\frac{1}{p}}$$

$$- \sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]} \; x$$

$$x \left\{[-A+(\tfrac{B-A}{2})\gamma(1-\alpha)] - \sqrt{[-A-1+(\tfrac{B-A}{2})\gamma(1-\alpha)][-A+1+(\tfrac{B-A}{2})\gamma(1-\alpha)]}\right\}^{1+\frac{1}{p}}$$

$$> o$$

which holds if

$$\beta > \frac{[-A+(\frac{B-A}{2})\gamma(1-\alpha)] - \sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)]}}{\left(\sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)]}\right)^p} \; x$$

$$\times \left\{\left(\frac{B-A}{2}\right)\gamma(1-\alpha)+\sqrt{(B-A)\gamma(1-\alpha)\left[-A+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]}\right\}^{p}.$$

Let $C_p(\alpha,\beta,\gamma,A,B)$ denote the right hand member of the above inequality. If $\beta = C_p(\alpha,\beta,\gamma,A,B)$, then we see that

$$a_o=\left(\frac{\left[-A+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]-\sqrt{\left[-A-1+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]\left[-A+1+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]}}{\beta}\right)^{\frac{1}{p}}.$$

This shows that

$$|z|<\left(\frac{\left[-A+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]-\sqrt{\left[-A-1+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]\left[-A+1+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]}}{\beta}\right)^{\frac{1}{p}}$$

is mapped on to a convex domain by f(z) provided $C_p(\alpha,\beta,\gamma,A,B) \leq \beta \leq 1$. To show that the estimate is sharp, we choose

$$f(z) = \frac{z}{\left\{1-[(B-A)\gamma+A]\beta z^p\right\}^{\frac{(B-A)\gamma(1-\alpha)}{[(B-A)\gamma+A]p}}}$$

so that $f(z) \in S_p^*(\alpha,\beta,\gamma,A,B)$ while

$$1 + \frac{zf''(z)}{f'(z)} = 0$$

when

$$z=\left(\frac{\left[-A+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]-\sqrt{\left[-A-1+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]\left[-A+1+\left(\frac{B-A}{2}\right)\gamma(1-\alpha)\right]}}{\beta}\right)^{\frac{1}{p}},$$

$o \leq \alpha < 1$, $o < \beta \leq 1$, $o < \gamma < \frac{-A}{(B-A)}$, $-1 \leq A < B \leq 1$, $o < B \leq 1$, so that $f(z)$ is not convex is any disc $|z| < R$ if R exceeds

$$\left(\frac{[-A+(\frac{B-A}{2})\gamma(1-\alpha)]-\sqrt{[-A-1+(\frac{B-A}{2})\gamma(1-\alpha)][-A+1+(\frac{B-A}{2})\gamma(1-\alpha)]}}{\beta}\right)^{\frac{1}{p}}.$$

Furthermore, for $o \leq \alpha < 1$, $o < \beta \leq 1$, $\gamma = \frac{-A}{(B-A)}$ and $-1 \leq A < B \leq 1$, $o < B \leq 1$, we ought to choose

$$f(z) = z \exp\left\{\frac{-A\beta(\alpha-1)}{p} z\right\}.$$

This completes the proof of the theorem.

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